

## RESEARCH ARTICLE

# Bayesian estimation of the mixture of exponentiated inverted Weibull distribution using noninformative and informative priors

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**Abstract:** In this study, Bayesian analysis of exponentiated inverted Weibull distribution (EIWD) is discussed. In particular, estimation of the scale parameter of the EIWD is focused whereas the shape parameter is assumed fixed. To derive the posterior distribution, uniform, Jeffreys, gamma and inverse Levy priors are used. Furthermore, to obtain Bayes estimates, the square error loss function (SELF), quadratic loss function (QLF), weighted loss function (WLF), precautionary loss function (PLF) and weighted balance loss function (WBLF) are considered. For comparison of the performance of different loss functions, the posterior risk is also calculated in this article. From application to the failure times of windshields dataset, results suggest that the uniform prior is a better prior than the Jeffreys prior, and WBLF is a suitable loss function for the estimation of the scale parameter of the mixture of EIWD. By comparing noninformative and informative priors, it is observed that the gamma prior has the minimum posterior risk.

**Keywords:** Bayes estimator, informative prior, loss function, mixture distribution, noninformative prior, posterior distribution.

## INTRODUCTION

There is a continuous effort in statistics to introduce flexible classes of probability distributions. Corderio *et al.* (2013) presented the idea of adding two-parameters to a continuous distribution. This idea of adding parameters was first introduced by Lehmann (1953) and later discussed by Nadarajah and Kotz (2006), which leads to a new class of exponentiated generalised distribution that can be interpreted as a double structure of Lehmann

alternatives. Several mathematical properties of this class have been obtained in literature, including the ordinary moments, generating function, mean deviation and order statistics. Moreover, the method of maximum likelihood estimation has also been investigated.

Datta (2013) compared the Weibull and exponentiated Weibull distributions and concluded that exponentiated Weibull distribution is more suitable than the Weibull distribution for wind speed data modelling. Elbatal and Muhammad (2014) derived the moment generating function and the  $r^{\text{th}}$  moment of the exponentiated inverted Weibull distribution (EIWD). Flaih *et al.* (2012) discussed some properties of the exponentiated distribution to failure times following bathtub pattern. Similarly, Gupta and Kundu (2001) studied a number of properties of the exponentiated exponential distribution. Moreover, Hassan *et al.* (2014) presented estimation of population parameters for the EIWD depending on equal and unequal group data. Lemonte and Corderio (2011) introduced the exponentiated generalised inverse Gaussian distribution and studied its properties.

To add flexibility to the existing probability distributions, finite mixture of lifetime models has received considerable attention of researchers. Many authors used mixture distributions in different real-life problems by using the frequentist and Bayesian estimation (Sultan *et al.*, 2007; Al-Hussaini & Hussein, 2012; Ali, 2014). AL-Hussaini and Hussein (2012) introduced a two-component finite mixture of the exponentiated

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model and studied its properties. Ali (2014) introduced and studied the mixture of inverse Rayleigh distribution and estimated its parameters by assuming informative and noninformative priors under different loss functions.

The Weibull distribution was introduced by a Swedish scientist Walodi Weibull in 1951 and has many applications in different fields such as medicine, ecology etc. (Rinne, 2008). This distribution and its various forms have also been considered in mixture modelling. For example, Ahmed *et al.* (2015) obtained Bayes estimators of the shape parameter of the exponentiated inverse Weibull distribution (EIWD) using a Bayesian technique under square error loss function, entropy loss function and the precautionary loss function. Aljuaid (2013) considered the complete and type-II censored data and obtained parameter estimates for EIWD. Li and Sedransk (1988) discussed a topological approach for mixture distributions. Sultan *et al.* (2007) introduced the mixture of inverse Weibull distribution and studied various properties, especially the identifiability property of the mixture model. Tahir *et al.* (2016) also discussed the mixture of exponential probability distributions.

The aim of this article is to introduce a two-component mixture of EIWD and estimate its parameters by the Bayesian approach. In particular, two different noninformative priors and five loss functions are considered to obtain the Bayes estimators. Moreover, the posterior risks under different loss functions to assess the performance of different Bayes estimators are also derived.

**METHODOLOGY**

**Two-component mixture model of the EIWD**

The cumulative distribution function (cdf) of the EIWD for a random variable X is defined as:

$$F(x; \theta, \beta) = (e^{-x^{-\beta}})^\theta, \quad x > \theta, \beta > 0, \quad \dots(1)$$

where  $\theta$  and  $\beta$  are the shape parameters. The probability density function (pdf) of the EIWD is given as:

$$f(x) = \theta\beta x^{-(\beta+1)} \left\{ \exp(-x^{-\beta}) \right\}^\theta \quad x, \theta, \beta > 0, \quad \dots(2)$$

To introduce a two-component mixture model of EIWD, it is supposed that the population consists of two types of subpopulations and  $w$  is the mixing proportion of the first type sub-population and  $(1-w)$  is the second type sub-population. The resulting mixture model is given as:

$$f(x) = wf_1(x) + (1-w)f_2(x), \quad 0 < w < 1. \quad \dots(3)$$

Let both the sub-populations follow the EIWD with a common shape parameter  $\beta$ , i.e.,

$$f_k(x) = \theta_k \beta x^{-(\beta+1)} \left\{ \exp(-x^{-\beta}) \right\}^{\theta_k}, \quad x, \theta_k, \beta > 0, \quad k = 1, 2 \quad \dots(4)$$

Then, the pdf of the mixture model is given as:

$$f(x) = w\theta_1\beta x^{-(\beta+1)} \left\{ \exp(-x^{-\beta}) \right\}^{\theta_1} + (1-w)\theta_2\beta x^{-(\beta+1)} \left\{ \exp(-x^{-\beta}) \right\}^{\theta_2},$$

Similarly, the cdf for the two-component mixture model is given by:

$$F(x) = wF_1(x) + (1-w)F_2(x) \quad 0 < w < 1$$

where

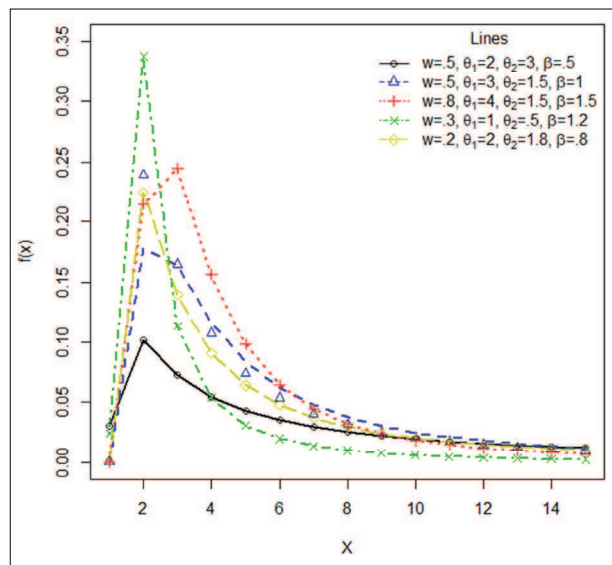
$$F_k(x) = (e^{-x^{-\beta}})^{\theta_k}, \quad x, \theta_k, \beta > 0 \quad k = 1, 2$$

Thus,

$$F(x) = w \left\{ \exp(-x^{-\beta}) \right\}^{\theta_1} + (1-w) \left\{ \exp(-x^{-\beta}) \right\}^{\theta_2};$$

$$x, \theta_1, \theta_2, \beta > 0, \quad 0 < w < 1 \quad \dots(5)$$

The graphical depiction of the mixture of EIWD for different parameter combinations is given in Figure 1.



**Figure 1:** Probability density function of the mixture of EIWD

Suppose that  $n$  elements are placed on a life testing experiment. Let  $r$  out of  $n$  be the elements having a lifetime  $[0, t_0]$ , where  $t_0$  is the pre-set termination time and the rest of the  $(n - r)$  elements are still functioning at the termination of the experiment. Further, let each subpopulation have  $r_k$  failed units and the rest as  $n = n_1 + n_2$ ,  $r = r_1 + r_2$  and  $k = 1, 2$ . Let  $x_{kj}$  denote the failure time of

the  $j^{th}$  element belonging to the  $k^{th}$  subpopulation, where  $j = 1, 2, \dots, r_k, 0 < x_{kj} \leq t_0$  and  $k = 1, 2$ . For such conditions, the likelihood function (Ali, 2014) is defined as:

$$L(\theta_1, \theta_2, \beta, w | \mathbf{x}) \propto \left\{ \prod_{j=1}^{n_1} f w_1(x_{1j}) \right\} \left\{ \prod_{j=1}^{n_2} (1-w) f_2(x_{2j}) \right\} \left\{ 1 - F(t_0) \right\}^{n-r}, \dots(6)$$

For the mixture of EWID, the likelihood function gets the form:

$$L(\theta_1, \theta_2, w | \mathbf{x}) \propto \left\{ \prod_{j=1}^{n_1} w \theta_1 \beta x_{1j}^{-(\beta+1)} \left\{ \exp(-x_{1j}^{-\beta}) \right\}^{\theta_1} \right\} \left\{ \prod_{j=1}^{n_2} (1-w) \theta_2 \beta x_{2j}^{-(\beta+1)} \left\{ \exp(-x_{2j}^{-\beta}) \right\}^{\theta_2} \right\} \\ \times \left[ 1 - \left\{ w \left\{ \exp(-t_0^{-\beta}) \right\}^{\theta_1} + (1-w) \left\{ \exp(-t_0^{-\beta}) \right\}^{\theta_2} \right\} \right]^{n-r}$$


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$$L(\theta_1, \theta_2, w | \mathbf{x}) \propto \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \theta_1^{r_1} \theta_2^{r_2} w^{r_1+k-l} (1-w)^{r_2+l} \\ \times \exp \left[ -\theta_1 \left\{ \sum_{j=1}^{n_1} w_{1j} + v(k-l) \right\} \right] \exp \left[ -\theta_2 \left\{ \sum_{j=1}^{n_2} w_{2j} + vl \right\} \right] \dots(7)$$

where  $w_{1j} = x_{1j}^{-\beta}, w_{2j} = x_{2j}^{-\beta}, v = t_0^{-\beta}$

**Posterior distribution using informative gamma prior**

In addition to the current/observed information, Bayesian methodology utilises the prior information and different types of priors are available in literature. Informative priors represent definite knowledge about a problem under study. In this section, gamma prior is considered as an informative prior. The prior distributions can be expressed

as:  $p(\theta_1) = \frac{b_1^{a_1}}{\Gamma(a_1)} \theta_1^{a_1-1} e^{-b_1 \theta_1}; p(\theta_2) = \frac{b_2^{a_2}}{\Gamma(a_2)} \theta_2^{a_2-1} e^{-b_2 \theta_2};$

and  $w \sim U(0,1)$ . The joint prior distribution of  $\theta_1, \theta_2$  and  $w$  is given in equation (8):

$$p(\theta_1, \theta_2, w) \propto \theta_1^{a_1-1} \theta_2^{a_2-1} e^{-b_1 \theta_1 - b_2 \theta_2}; \theta_1, \theta_2 > 0, 0 < w < 1 \dots(8)$$

where  $a_1, a_2, b_1$  and  $b_2$  are the parameters of the prior distribution and known as the hyperparameters. The joint posterior distribution of  $\theta_1, \theta_2$  and  $w$  is calculated by the celebrated Bayes theorem as:

$$p(\theta_1, \theta_2, w | \mathbf{x}) = \frac{p(\theta_1, \theta_2, w) L(\theta_1, \theta_2, w | \mathbf{x})}{\int_0^1 \int_0^\infty \int_0^\infty p(\theta_1, \theta_2, w) L(\theta_1, \theta_2, w | \mathbf{x}) d\theta_1 d\theta_2 dw}$$

$$p(\theta_1, \theta_2, w | \mathbf{x}) \propto \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \theta_1^{r_1+a_1-1} \theta_2^{r_2+a_2-1} w^{r_1+k-l} (1-w)^{r_2+l}$$

$$\begin{aligned} & \times \exp \left[ -\theta_1 \left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\} \right] \\ & \times \exp \left[ -\theta_2 \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\} \right]; \quad \theta_1, \theta_2 > 0, \\ p(\theta_1, \theta_2, w | \mathbf{x}) &= \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \\ & \theta_1^{r_1+a_1-1} \theta_2^{r_2+a_2-1} w^{r_1+k-l} (1-w)^{r_2+l} \\ & \times \exp \left[ -\theta_1 \left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + (k-l)v \right\} - \right. \\ & \left. \theta_2 \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + (l)v \right\} \right]; \quad \theta_1, \theta_2 > 0, 0 < w < 1, \end{aligned} \tag{09}$$

where

$$\begin{aligned} C_3 &= \int_0^1 \int_0^1 \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \\ & \theta_1^{r_1+a_1-1} \theta_2^{r_2+a_2-1} w^{r_1+k-l} (1-w)^{r_2+l} \\ & \times \exp \left[ -\theta_1 \left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\} - \right. \\ & \left. \theta_2 \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\} \right] dw d\theta_2 d\theta_1 \\ C_3 &= \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \\ & \beta(r_1+k-l+1, r_2+l+1) \times \\ & \frac{\Gamma(r_1+a_1)\Gamma(r_2+a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}} \end{aligned}$$

and  $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$  is the beta function. In the next section, marginal posterior distributions of each parameter are derived.

### The marginal posterior distributions using GP

In this section, the derivation of the marginal posterior distribution of each parameter by integrating out the nuisance parameter is discussed. The marginal posterior distribution of  $\theta_1$  using GP is obtained as:

$$\begin{aligned} p(\theta_1 | x) &= \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \\ & \beta(r_1+k-l+1, r_2+l+1) \\ & \times \frac{\Gamma(r_2+a_2)}{\left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}} \theta_1^{r_1+a_1-1} \\ & \exp \left[ -\theta_1 \left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\} \right]; \quad \theta_1 > 0 \end{aligned} \tag{10}$$

The marginal posterior distribution of  $\theta_2$  is given by:

$$\begin{aligned} p(\theta_2 | x) &= \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \\ & \beta(r_1+k-l+1, r_2+l+1) \\ & \times \frac{\Gamma(r_1+a_1)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)}} \theta_2^{r_2+a_2-1} \\ & \exp \left[ -\theta_2 \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\} \right]; \quad \theta_2 > 0 \end{aligned} \tag{11}$$

The marginal posterior distribution of  $w$  is given by:

$$\begin{aligned} p(w | x) &= \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \\ & \times \frac{\Gamma(r_1+a_1)\Gamma(r_2+a_2)w^{r_1+k-l}(1-w)^{r_2+l}}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}; \\ & 0 < w < 1 \end{aligned} \tag{12}$$

In the next section, the derivation of the Bayes estimators is discussed.

**Bayesian estimation**

The aim of this section is to derive Bayes estimators of the mixture of EWID and for this, five different loss functions, namely, square error loss function (SELF), quadratic loss function (QLF), weighted loss function (WLF), precautionary loss function (PLF) and weighted balance loss function (WBLF) are considered. To this end, the derivation of Bayes estimators and their posterior

risks assuming the GP under the above-mentioned loss functions is presented. The Bayes estimator  $d^*$  is derived by minimising the posterior expected loss, defined as  $\mathfrak{R}(d^*) = E_{\theta|x} \{L(\theta, d^*)\}$ , where  $L(\theta, d^*)$  is the loss incurred estimating  $\theta$  by  $d^*$ . For a given prior, the general form of the Bayes estimators (BEs) and posterior risks (PRs) under SELF, QLF, WLF, PLF and WBLF are given in Table 1 (Ali *et al.*, 2014).

**Table 1:** Bayes estimators and posterior risks under SELF, PLF and DLF

Loss function	Bayes estimators (BEs)	Posterior risks (PRs)
$SELF = L(\theta, d^*) = (\theta - d^*)^2$	$d^* = E_{\theta x}(\theta)$	$\mathfrak{R}(d^*) = E_{\theta x}(\theta^2) - \{E_{\theta x}(\theta)\}^2$
$QLF = L(\theta, d^*) = \left(\frac{\theta - d^*}{\theta}\right)^2$	$d^* = \{E_{\theta x}(\theta^{-2})\}^{-1} E_{\theta x}(\theta^{-1})$	$\mathfrak{R}(d^*) = 1 - \{E_{\theta x}(\theta^{-2})\}^{-1} \{E_{\theta x}(\theta^{-1})\}^2$
$WLF = L(\theta, d^*) = \frac{(\theta - d^*)^2}{\theta}$	$d^* = \{E_{\theta x}(\theta^{-1})\}^{-1}$	$\mathfrak{R}(d^*) = E_{\theta x}(\theta) - \{E_{\theta x}(\theta^{-1})\}^{-1}$
$PLF = L(\theta, d^*) = \frac{(\theta - d^*)^2}{d^*}$	$d^* = \{E_{\theta x}(\theta^2)\}^{\frac{1}{2}}$	$\mathfrak{R}(d^*) = 2 \{E_{\theta x}(\theta^2)\}^{\frac{1}{2}} - 2E_{\theta x}(\theta)$
$WBLF = L(\theta, d^*) = \left(\frac{\theta - d^*}{d^*}\right)^2$	$d^* = \{E_{\theta x}(\theta)\}^{-1} E_{\theta x}(\theta^2)$	$\mathfrak{R}(d^*) = 1 - \{E_{\theta x}(\theta)\}^2 \{E_{\theta x}(\theta^2)\}^{-1}$

Taking  $d^* = \theta_1^*$ ,  $\theta_2^*$  and  $w^*$  for estimating the parameters  $\theta = \theta_1, \theta_2$  and  $w_1$ , the expressions for the Bayes estimators and posterior risks are given in the following subsections:

**Bayes estimators and the Bayes risks under SELF**

The Bayes estimator of the parameter  $\theta_1$  under SELF using equation (10) is obtained as:

$$\theta_1^* = \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1 + 1) \Gamma(r_2 + a_2)}{\left\{b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l)\right\}^{r_1+a_1+1} \left\{b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l)\right\}^{r_2+a_2}} \dots(13)$$

The Bayes risk of  $\theta_1$  under SELF is obtained as:

$$\mathfrak{R}(\theta_1^*) = \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \left[ \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1 + 2) \Gamma(r_2 + a_2)}{\left\{b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l)\right\}^{(r_1+a_1+2)} \left\{b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l)\right\}^{(r_2+a_2)}} - \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1 + 1) \Gamma(r_2 + a_2)}{\left\{b_1 + \sum_{j=1}^{r_1} w_{1j}^{-1} + v(k-l)\right\}^{(r_1+a_1+1)} \left\{b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l)\right\}^{(r_2+a_2)}} \right]^2 \dots(14)$$

The Bayes estimator of the parameter  $\theta_2$  under SELF using equation (11) is obtained as:

$$\theta_2^* = \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + a_2 + 1)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2+1)}} \dots(15)$$

The Bayes risk of  $\theta_2$  under SELF is obtained as:

$$\mathfrak{R}(\theta_2^*) = \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + a_2 + 3)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2+3)}} \left[ \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + a_2 + 1)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2+1)}}} \right]^2 \dots(16)$$

The Bayes estimator of the parameter  $w$  under SELF using equation (12) is obtained as:

$$w^* = \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k + 2, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}} \dots(17)$$

The Bayes risk of  $w$  under SELF is obtained as:

$$\mathfrak{R}(w^*) = \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k + 3, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}} \left[ \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k + 2, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}} \right]^2 \dots(18)$$

**Bayes estimators and the Bayes risks under QLF**

The Bayes estimator of the parameter  $\theta_1$  under QLF using equation (10) is obtained as :

$$\theta_1^* = \frac{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1 - 1) \Gamma(r_2 + a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1-1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}}{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1 - 2) \Gamma(r_2 + a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1-2)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}}} \dots(19)$$

The Bayes risk of  $\theta_1$  under QLF is obtained as:

$$\mathfrak{R}(\theta_1^*) = 1 - \frac{\frac{1}{C_3} \left[ \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_1+a_1-1) \Gamma(r_2+a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1-1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}} \right]}{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_1+a_1-2) \Gamma(r_2+a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1-2)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}}$$

...(20)

The Bayes estimator of the parameter  $\theta_2$  under QLLF using equation (11) is obtained as:

$$\theta_2^* = \frac{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_1+a_1) \Gamma(r_2+a_2-1)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2-1)}}}{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_1+a_1) \Gamma(r_2+a_2-2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2-2)}}}$$

...(21)

The Bayes risk of  $\theta_2$  under QLF is obtained as:

$$\mathfrak{R}(\theta_2^*) = 1 - \frac{\frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_1+a_1) \Gamma(r_2+a_2-1)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{r_1+a_1} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{r_2+a_2-1}}}{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_1+a_1) \Gamma(r_2+a_2-1)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{r_1+a_1} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{r_2+a_2-1}}}$$

...(22)

The Bayes estimator of the parameter  $w$  under QLF using equation (12) is obtained as:

$$w^* = \frac{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l, r_2+l+1) \Gamma(r_1+a_1) \Gamma(r_2+a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}}{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l-1, r_2+l+1) \Gamma(r_1+a_1) \Gamma(r_2+a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}}$$

...(23)

The Bayes risk of  $w$  under QLF is obtained as:



$$\mathfrak{R}(\theta_1^*) = 1 - \frac{\frac{1}{C_3} \left[ \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}} \right]^2}{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l - 1, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}}} \quad \dots(24)$$

**Bayes estimators and the Bayes risks under precautionary loss function**

The Bayes estimator of the parameter  $\theta_1$  under PLF using equation (10) is obtained as:

$$\theta_1^* = \left[ \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1 + 2) \Gamma(r_2 + a_2)}{\left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1+2)} \left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}} \right]^{\frac{1}{2}} \quad \dots(25)$$

The Bayes risk of  $\theta_1$  under PLF is obtained as:

$$\mathfrak{R}(\theta_1^*) = \left[ \frac{4}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1 + 2) \Gamma(r_2 + a_2)}{\left[ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right]^{(r_1+a_1+2)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}} \right]^{\frac{1}{2}} - \frac{2}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1 + 1) \Gamma(r_2 + a_2)}{\left[ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right]^{(r_1+a_1+1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}} \quad \dots(26)$$

The Bayes estimator of the parameter  $\theta_2$  under PLF using equation (11) is obtained as:

$$\theta_2^* = \left[ \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + a_2 + 2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2+2)}} \right]^{\frac{1}{2}} \quad \dots(27)$$

The Bayes risk of  $\theta_2$  under PLF is obtained as:



$$\mathfrak{R}(\theta_2^*) = \left[ \frac{4}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_1+a_1) \Gamma(r_2+a_2+2)}{\left[ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right]^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2+2)}} \right]^{\frac{1}{2}} - \frac{2}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_1+a_1) \Gamma(r_2+a_2+1)}{\left[ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right]^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2+1)}} \dots(28)$$

The Bayes estimator of the parameter  $w$  under PLF using equation (12) is obtained as:

$$w^* = \left[ \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k+3, r_2+l+1) \Gamma(r_1+a_1) \Gamma(r_2+a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}} \right]^{\frac{1}{2}} \dots(29)$$

The Bayes risk of  $w$  under PLF is obtained as:

$$\mathfrak{R}(w^*) = \left[ \frac{4}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+3, r_2+l+1) \Gamma(r_1+a_1) \Gamma(r_2+a_2)}{\left[ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right]^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}} \right]^{\frac{1}{2}} - \frac{2}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+2, r_2+l+1) \Gamma(r_1+a_1) \Gamma(r_2+a_2)}{\left[ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right]^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}} \dots(30)$$

**Bayes estimators and the Bayes risks under weighted balance loss function**

The Bayes estimator of the parameter  $\theta_1$  under WBLF using equation (10) is obtained as:

$$\theta_1^* = \frac{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_1+a_1+2) \Gamma(r_2+a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1+2)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}}{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_1+a_1+1) \Gamma(r_2+a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1+1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}} \dots(31)$$

The Bayes risk of  $\theta_1$  under WBLF is obtained as:

$$\mathfrak{R}(\theta_1^*) = 1 - \frac{\frac{1}{C_3} \left[ \sum_{k=1}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1 + 1) \Gamma(r_2 + a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1+1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}} \right]^2}{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1 + 2) \Gamma(r_2 + a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1+2)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}}$$

...(32)

The Bayes estimator of the parameter  $\theta_2$  under WBLF using equation (11) is obtained as:

$$\theta_2^* = \frac{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + a_2 + 2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2+2)}}}{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + 1)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+1)}}}$$

...(33)

The Bayes risk of  $\theta_2$  under WBLF is obtained as:

$$\mathfrak{R}(\theta_2^*) = 1 - \frac{\frac{1}{C_3} \left[ \sum_{k=1}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + a_2 + 1)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2+1)}} \right]^2}{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 1, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + a_2 + 2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + w(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2+2)}}}$$

...(34)

The Bayes estimator of the parameter  $w$  under WBLF using equation (12) is obtained as:

$$w^* = \frac{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 3, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}}{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1 + k - l + 2, r_2 + l + 1) \Gamma(r_1 + a_1) \Gamma(r_2 + a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}}$$

...(35)

The Bayes risk of  $w$  under WBLF is obtained as:

$$\mathfrak{R}(w^*) = 1 - \frac{\frac{1}{C_3} \left[ \sum_{k=1}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+2, r_2+l+1) \Gamma(r_1+a_1) \Gamma(r_2+a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}} \right]}{\sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+3, r_2+l+1) \Gamma(r_1+a_1) \Gamma(r_2+a_2)}{\left\{ b_1 + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)} \left\{ b_2 + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)}}}} \quad \dots(36)$$

**Bayes estimators and the Bayes risks under weighted loss function**

The Bayes estimator of the parameter  $\theta_1$  under WLF using equation (10) is obtained as:

$$\theta_1^* = \left[ \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_2+a_2) \Gamma(r_1+a_1-1)}{\left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)} \left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1-1)}}} \right]^{-1} \quad \dots(37)$$

The Bayes risk of  $\theta_1$  under WLF is obtained as:

$$\mathfrak{R}(\theta_1^*) = \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_2+a_2) \Gamma(r_1+a_1+1)}{\left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)} \left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1+1)}}} \quad \dots(38)$$

The Bayes estimator of the parameter  $\theta_2$  under WLF using equation (11) is obtained as:

$$\theta_2^* = \left[ \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_2+a_2-1) \Gamma(r_1+a_1)}{\left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2-1)} \left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)}}} \right]^{-1} \quad \dots(39)$$

The Bayes risk of  $\theta_2$  under WLF is obtained as:

$$\mathfrak{R}(\theta_2^*) = \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_2+a_2+1) \Gamma(r_1+a_1)}{\left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2+1)} \left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)}}} - \left[ \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+1, r_2+l+1) \Gamma(r_2+a_2-1) \Gamma(r_1+a_1)}{\left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2-1)} \left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)}}} \right]^{-1} \quad \dots(40)$$

The Bayes estimator of the parameter  $w$  under WLF using equation (12) is obtained as:

$$w^* = \left[ \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l, r_2+l+1) \Gamma(r_2+a_2) \Gamma(r_1+a_1)}{\left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)} \left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)}}} \right]^{-1} \quad \dots(41)$$

The Bayes risk of  $w$  under WLF is obtained as:

$$\mathfrak{R}(w^*) = \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l+2, r_2+l+1) \Gamma(r_2) \Gamma(r_1)}{\left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2)} \left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1)}}} - \left[ \frac{1}{C_3} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \frac{\beta(r_1+k-l, r_2+l+1) \Gamma(r_2+a_2) \Gamma(r_1+a_1)}{\left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+a_2)} \left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+a_1)}}} \right]^{-1} \quad \dots(42)$$

**Posterior distribution assuming inverse Levy prior**

In addition to the GP, inverse Levy prior (ILP) is taken as an

informative prior. It is defined as:  $p(\theta_1) = \left(\frac{b_1}{2\pi}\right)^{\frac{1}{2}} \theta_1^{-\frac{1}{2}} e^{-\frac{b_1}{2}\theta_1}$ ;

$\theta_1 > 0, \quad p(\theta_2) = \left(\frac{b_2}{2\pi}\right)^{\frac{1}{2}} \theta_2^{-\frac{1}{2}} e^{-\frac{b_2}{2}\theta_2}, \quad \theta_2 > 0$  whereas

$p(w) = 1; \quad 0 < w < 1$ , Thus, the joint prior distribution of  $\theta_1, \theta_2$  and  $w$  is given by:

$$p(\theta_1, \theta_2, w) \propto \theta_1^{-\frac{1}{2}} \theta_2^{-\frac{1}{2}} e^{-\frac{b_1}{2}\theta_1 - \frac{b_2}{2}\theta_2};$$

$$\theta_1, \theta_2 > 0, \quad 0 < w < 1 \quad \dots(43)$$

The joint posterior distribution of  $\theta_1, \theta_2$  and  $w$  is given below:

$$p(\theta_1, \theta_2, w|x) \propto \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l}$$

$$\theta_1^{r_1-\frac{1}{2}} \theta_2^{r_2-\frac{1}{2}} w^{r_1+k-l} (1-w)^{r_2+l}$$

$$\times \exp \left[ -\theta_1 \left\{ \frac{b_1}{2} + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\} - \theta_2 \left\{ \frac{b_2}{2} + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\} \right]$$

$$\theta_1, \theta_2 > 0, \quad 0 < w < 1 \quad \dots(44)$$

The marginal posterior distribution of  $\theta_1$  can be written as:

$$p(\theta_1|x) = \frac{1}{C_4} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \beta(r_1+k-l+1, r_2+l+1)$$

$$\times \frac{\Gamma(r_2+\frac{1}{2})}{\left\{ \frac{b_2}{2} + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+\frac{1}{2})}}$$

$$\theta_1^{r_1-\frac{1}{2}} \exp \left[ -\theta_1 \left\{ \frac{b_1}{2} + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\} \right]; \quad \theta_1 > 0 \quad \dots(45)$$

where  $C_4 = \int_0^1 \int_0^1 \int_0^\infty \int_0^\infty \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l}$

$$\theta_1^{r_1-\frac{1}{2}} \theta_2^{r_2-\frac{1}{2}} w^{r_1+k-l} (1-w)^{r_2+l}$$

$$\times \exp \left[ -\theta_1 \left\{ \frac{b_1}{2} + \sum_{j=1}^{r_1} w_{1j} + (k-l)v \right\} - \theta_2 \left\{ \frac{b_2}{2} + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\} \right] dw d\theta_2 d\theta_1$$

$$= \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \beta(r_1 + k - l + 1, r_2 + l + 1) \\ \times \frac{\Gamma(r_1 + \frac{1}{2})\Gamma(r_2 + \frac{1}{2})}{\left\{ \frac{b_1}{2} + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1 + \frac{1}{2})} \left\{ \frac{b_2}{2} + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2 + \frac{1}{2})}}$$

The marginal posterior distribution of  $\theta_2$  is obtained as:

$$p(\theta_2|x) = \frac{1}{C_4} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \beta(r_1 + k - l + 1, r_2 + l + 1) \\ \times \frac{\Gamma(r_1 + \frac{1}{2})}{\left\{ \frac{b_1}{2} + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1 + \frac{1}{2})}} \theta_1^{r_2 - \frac{1}{2}} \\ \exp \left[ -\theta_2 \left\{ \frac{b_2}{2} + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\} \right] \quad \dots(46)$$

The marginal posterior distribution of  $w$  is obtained as:

$$p(w|x) = \frac{1}{C_4} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \\ \times \frac{\Gamma(r_1 + \frac{1}{2})\Gamma(r_2 + \frac{1}{2})w^{r_1+k-l}(1-w)^{r_2+l}}{\left\{ \frac{b_1}{2} + \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1 + \frac{1}{2})} \left\{ \frac{b_2}{2} + \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2 + \frac{1}{2})}}$$

... (47)

$0 < w < 1$

Assuming  $a_1 = a_2 = 0.5$ , and  $b_1 = b_1/2, b_2 = b_2/2$  in equations (13) to (42), one can obtain the Bayes estimators and their respective posterior risk under different loss functions for the JP.

**Posterior distribution under noninformative priors**

In this section, the derivation of the posterior distributions assuming noninformative priors is discussed. Noninformative priors are priors which are less informative than informative priors. The two most

commonly used noninformative priors are the Jeffreys prior (JP) and uniform prior (UP) (Ali *et al.*, 2012).

**JP:** The Jeffreys prior is given by  $p(\theta_1) \propto \frac{1}{\theta_1}; \theta_1 > 0, p(\theta_2) \propto \frac{1}{\theta_2}; \theta_2 > 0$ , while the uniform prior is assumed for  $w$ , i.e.,  $p(w) = 1; 0 < w < 1$ , then the joint prior distribution of  $\theta_1, \theta_2$  and  $w$  is:

$$p(\theta_1, \theta_2, w) \propto \frac{1}{\theta_1 \theta_2}; \theta_1, \theta_2 > 0, 0 < w < 1, \quad \dots(48)$$

Thus, the joint posterior distribution of  $\theta_1, \theta_2$  and  $w$  is given by:

$$p(\theta_1, \theta_2, w|x) \propto \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \\ |\theta_1^{r_1-1} \theta_2^{r_2-1} w^{r_1+k-l} (1-w)^{r_2+l} \times \exp \left[ -\theta_1 \left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\} \right. \\ \left. - \theta_2 \left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\} \right], \theta_1, \theta_2 > 0, 0 < w < 1$$

$$p(\theta_1, \theta_2, w|x) = \frac{1}{C_2} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \\ \theta_1^{r_1-1} \theta_2^{r_2-1} w^{r_1+k-l} (1-w)^{r_2+l} \times \exp \left[ -\theta_1 \left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\} \right. \\ \left. - \theta_2 \left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\} \right], \theta_1, \theta_2 > 0, 0 < w < 1 \quad \dots(49)$$

where  $C_2 = \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l}$

$$\beta(r_1 + k - l + 1, r_2 + l + 1)^*$$

$$\frac{\Gamma(r_1)\Gamma(r_2)}{\left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1)} \left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2)}}$$

The marginal posterior distribution of  $\theta_1$  using JP is obtained from equation (49) and is given by:

$$p(\theta_1|\mathbf{x}) = \frac{1}{C_2} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \beta(r_1 + k - l + 1, r_2 + l + 1) \times \frac{\Gamma(r_2)}{\left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{r_2}} \theta_1^{r_1-1} \exp \left[ -\theta_1 \left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\} \right]; \theta_1 > 0 \dots(50)$$

The marginal posterior distribution of  $\theta_2$  is given by:

$$p(\theta_2|\mathbf{x}) = \frac{1}{C_2} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \beta(r_1 + k - l + 1, r_2 + l + 1) \times \frac{\Gamma(r_1)}{\left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1)}} \theta_2^{r_2-1} \exp \left[ -\theta_2 \left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\} \right]; \theta_2 > 0 \dots(51)$$

The marginal posterior distribution of  $w$  is given below:

$$p(w|\mathbf{x}) = \frac{1}{C_2} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \times \frac{\Gamma(r_1)\Gamma(r_2)w^{r_1+k-l}(1-w)^{r_2+l}}{\left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1)} \left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2)}}} 0 < w < 1 \dots(52)$$

Assuming  $a_1 = a_2 \rightarrow 0$  and  $b_1 = b_2 \rightarrow 0$  in equations (13) to (42), the Bayes estimators and the respective posterior risks under different loss functions for the JP can be obtained.

**UP:** Now, considering the uniform prior, i.e.,  $\theta_1, \theta_2 \sim U(0, \infty)$  and  $w \sim U(0,1)$ , the joint prior distribution of  $\theta_1, \theta_2$  and  $w$  is given by:

$$p(\theta_1, \theta_2, w) \propto 1; \quad \theta_1, \theta_2 > 0, \quad 0 < w < 1 \dots(53)$$

The joint posterior distribution of  $\theta_1, \theta_2$  and  $w$  is given by:

$$p(\theta_1, \theta_2, w|\mathbf{x}) \propto \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \theta_1^{r_1} \theta_2^{r_2} w^{r_1+k-l} (1-w)^{r_2+l} \times \exp \left[ -\theta_1 \left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\} - \theta_2 \left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\} \right], \theta_1, \theta_2 > 0, 0 < w < 1$$

$$p(\theta_1, \theta_2, w|\mathbf{x}) = \frac{1}{C_1} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \theta_1^{r_1} \theta_2^{r_2} w^{r_1+k-l} (1-w)^{r_2+l} \times \exp \left[ -\theta_1 \left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\} - \theta_2 \left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\} \right], \theta_1, \theta_2 > 0, 0 < w < 1, \dots(54)$$

where

$$C_1 = \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l}$$

$$\beta(r_1 + k - l + 1, r_2 + l + 1) \frac{\Gamma(r_1 + 1)\Gamma(r_2 + 1)}{* \left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+1)} \left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+1)}}$$

The marginal posterior distribution of  $\theta_1$  using the UP is obtained from equation (54):

$$p(\theta_1|\mathbf{x}) = \frac{1}{C_1} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \beta(r_1 + k - l + 1, r_2 + l + 1) \frac{\Gamma(r_2 + 1)}{\left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+1)}} \theta_1^{r_1} \exp \left[ -\theta_1 \left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+1)} \right]; \theta_1 > 0 \dots(55)$$

The marginal posterior distribution of  $\theta_2$  is given by:

$$p(\theta_2|x) = \frac{1}{C_1} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \beta(r_1 + k - l + 1, r_2 + l + 1) \times \frac{\Gamma(r_1 + 1)}{\left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+1)}} \theta_2^{r_2} \exp \left[ -\theta_2 \left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\} \right]; \theta_2 > 0 \quad \dots(56)$$

The marginal posterior distribution of  $w$  is obtained as:

$$p(w|x) = \frac{1}{C_1} \sum_{k=0}^{n-r} \sum_{l=0}^k (-1)^k \binom{n-r}{k} \binom{k}{l} \times \frac{\Gamma(r_1 + 1)\Gamma(r_2 + 1)w^{r_1+k-l} (1-w)^{r_2+l}}{\left\{ \sum_{j=1}^{r_1} w_{1j} + v(k-l) \right\}^{(r_1+1)} \left\{ \sum_{j=1}^{r_2} w_{2j} + v(l) \right\}^{(r_2+1)}}, \quad 0 < w < 1. \quad \dots(57)$$

Assuming  $a_1 = a_2 = 1$  and  $b_1 = b_2 \rightarrow 0$  in equations (13) to

(42), the Bayes estimators and their respective posterior risks under different loss functions for the UP can be obtained.

## RESULTS AND DISCUSSION

### Real life data application

Murthy *et al.* (2004) presented a real-life dataset, which is given in Table 2, consisting of 88 observations on the failure times of windshields. The descriptive summary of the data is given in Table 3 while the Bayes estimates and their posterior risks under different loss functions are given in Tables 4 and 5.

**Table 2:** Real life dataset

0.040	0.301	0.309	0.557	0.943	1.070	1.124	1.248
1.281	1.281	1.303	1.432	1.480	1.505	1.506	1.568
1.615	1.619	1.652	1.652	1.757	1.795	1.866	1.876
1.899	1.911	1.912	1.914	1.981	2.010	2.038	2.085
2.085	2.097	2.135	2.154	2.190	2.194	2.223	2.224
2.229	2.300	2.324	2.349	2.385	2.481	2.610	2.625
2.632	2.646	2.661	2.688	2.823	2.890	2.902	2.934
2.962	2.964	3.000	3.103	3.114	3.117	3.166	3.344
3.376	3.385	3.443	3.467	3.478	3.578	3.595	3.699
3.779	3.924	4.035	4.121	4.167	4.240	4.255	4.278
4.305	4.376	4.449	4.485	4.570	4.602	4.663	4.694

**Table 3:** Descriptive summary of real data

n	Mean	SD	SE	Mode	Median	Kurtosis	Skewness	Minimum	Maximum
88	2.58	1.12	0.12	1.28	2.36	-0.62	0.12	0.04	4.69

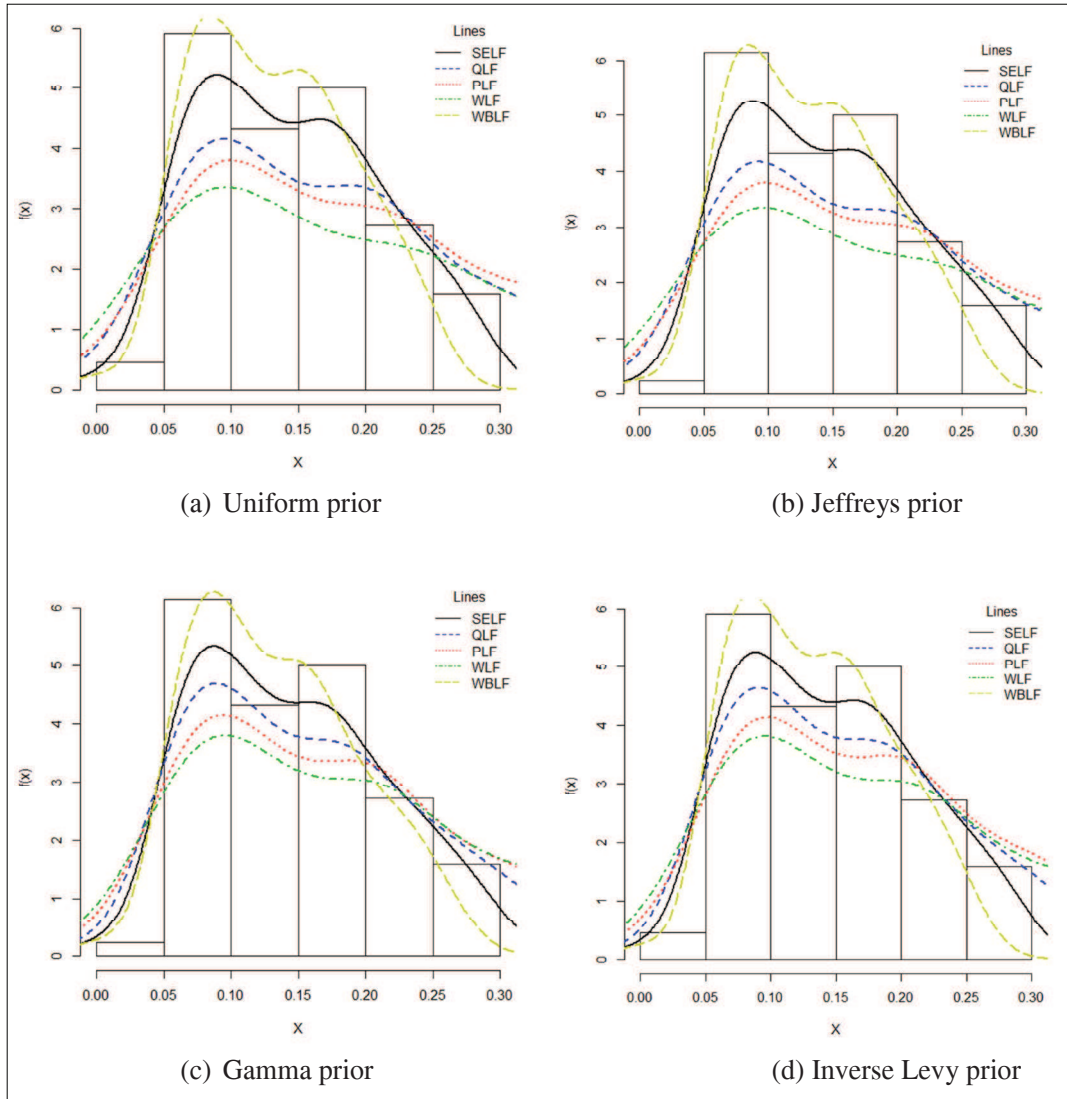
**Table 4:** BEs and BRs for the real dataset using non-informative priors and 20 % censoring rate

Loss functions	Uniform prior			Jeffreys prior		
	BEs ( $\theta_1, \theta_2, \beta = 1, w$ )	PRs		BEs ( $\theta_1, \theta_2, \beta = 1, w$ )	PRs	
SELF	1.6398, 2.0559, 0.3875	0.0844, 0.0952, 0.0035		1.5897, 2.0112, 0.3867	0.0819, 0.0932, 0.0035	
QLF	1.5368, 1.9631, 0.3784	0.0325, 0.0231, 0.0237		1.4864, 1.9183, 0.3681	0.9664, 0.0236, 0.0249	
PLF	1.6654, 2.0789, 0.3852	0.0510, 0.0461, 0.0237		1.6152, 2.0342, 0.3781	0.0511, 0.0461, 0.0242	
WLF	1.5719, 2.0088, 0.3920	0.0304, 0.0251, 0.0089		1.5695, 1.9855, 0.3911	0.9701, 0.0292, 0.0089	
WBLF	1.6913, 2.1022, 0.3965	0.0304, 0.0220, 0.0226		1.6412, 2.0575, 0.3957	0.0314, 0.0225, 0.0227	



This dataset is considered because failure times follow the Weibull distribution (Murthy *et al.*, 2004) and EIWD is a more flexible generalisation of the ordinary Weibull distribution. Thus, it is more appropriate to use the dataset for further analysis. Using the above dataset,

BEs and PRs have been calculated under five different loss functions (SELF, QLF, PLF, WLF and WBLF) and different types of non-informative and informative priors. The resulting study is tabulated in Tables 4 and 5 while the graphical depiction is given in Figure 2.



**Figure 2:** A comparison of observed and theoretical densities using UP, JP, GP and ILP

From Table 4, by comparing Bayes risks of different loss functions, it is obvious that WBLF has the minimum posterior risk as compared to the other loss functions. Therefore, the WBLF is the most suitable loss function for the Bayesian estimation of the scale parameter of the EIWD. The above results also suggest that the performance of the uniform prior is better than the Jeffreys prior. Similar to Table 4, Table 5 suggests that

the gamma prior is better than the inverse Levy prior and WLF has the minimum risk as compared to the rest. It is also observed that informative priors are more preferable than the noninformative priors because of minimum risks, and gamma prior has the least risk among the assumed priors. It is worth mentioning that the hyperparameters of the informative priors are selected to represent mild information than the noninformative case. A graphical

depiction of the observed and theoretical densities have been given in Figure 2. The interpretation of Figure 2

is the same as the interpretation of the results given in Tables 4 and 5.

**Table 5:** BEs and BRs for the real dataset using informative priors and 20 % censoring rate

Loss functions	Gamma prior			Inverse Levy prior		
	BEs ( $\theta_1, \theta_2, \beta = 1, w$ )			PRs		
SELF	1.5254, 2.0112, 0.3853	0.0721, 0.0932, 0.0035	1.6041, 2.0309, 0.3869	0.0820, 0.0939, 0.0035		
QLF	1.4307, 1.9183, 0.3667	0.0279, 0.0236, 0.0249	1.5017, 1.9382, 0.3684	0.0330, 0.9766, 0.0248		
PLF	1.5489, 1.9513, 0.3898	0.0469, 0.0426, 0.0089	1.6295, 2.0539, 0.3915	0.0507, 0.0460, 0.0089		
WLF	1.5074, 1.9325, 0.3745	0.0234, 0.0287, 0.0255	1.5156, 1.9728, 0.3651	0.0321, 0.9663, 0.0237		
WBLF	1.5727, 1.9729, 0.3943	0.0221, 0.0217, 0.0228	1.6553, 2.0772, 0.3959	0.0309, 0.0223, 0.0227		

## CONCLUSION

A two-component mixture model of the EIWD is considered in this article. In addition to the joint posterior distribution, the marginal posterior distributions for  $\theta_1, \theta_2$  and  $w$  are also derived assuming the uniform, Jeffreys, gamma, and inverse Levy priors. Moreover, to derive the Bayes estimates and their posterior risks, five different loss functions, namely, SELF, QLF, PLF, WLF, and WBLF have been considered. A real-life dataset is used for the selection of an appropriate loss function and of a prior. The results showed that the WBLF is a better loss function while the uniform prior is suitable for estimating  $\theta_1, \theta_2$  and  $w$ , as the posterior risks for the Bayes estimators of  $\theta_1, \theta_2$  and  $w$  under WBLF were observed minimum assuming this prior. Similarly, the gamma prior is more informative than the inverse Levy prior. However, it is also observed that the informative priors have the minimum posterior risk as compared to the assumed loss functions. In future, Bayesian analysis for more than two-component mixture of the EIWD can be considered.

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